

# Supersymmetry algebra cohomology II: Primitive elements in 2 and 3 dimensions

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## Abstract

The primitive elements of the supersymmetry algebra cohomology as defined in a companion paper are computed exhaustively for standard supersymmetry algebras in dimensions  $D = 2$  and  $D = 3$ , for all signatures  $(t, D - t)$  and all numbers  $N$  of sets of supersymmetries.

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# 1 Introduction

This paper relates to supersymmetry algebra cohomology as defined in [1], for supersymmetry algebras in dimensions  $D = 2$  and  $D = 3$  of translational generators  $P_a$  ( $a = 1, \dots, D$ ) and supersymmetry generators  $Q_{\underline{\alpha}}^i$  of the form

$$[P_a, P_b] = 0, \quad [P_a, Q_{\underline{\alpha}}^i] = 0, \quad \{Q_{\underline{\alpha}}^i, Q_{\underline{\beta}}^j\} = -i \delta^{ij} (\Gamma^a C^{-1})_{\underline{\alpha}\underline{\beta}} P_a \quad (1.1)$$

where  $\delta^{ij}$  denotes the Kronecker delta for  $N$  sets of supersymmetries,<sup>1</sup>

$$\delta^{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases} \quad (1.2)$$

and  $C$  is a charge conjugation matrix fulfilling in all cases under study

$$\forall a: \quad C \Gamma^a C^{-1} = -\Gamma^{a\top} \quad (1.3)$$

and

$$C^\top = -C. \quad (1.4)$$

The object of this paper is the determination of the primitive elements of the supersymmetry algebra cohomology for supersymmetry algebras (1.1) in  $D = 2$  and  $D = 3$  dimensions, for all numbers  $N$  of sets of supersymmetries and all signatures  $(t, D - t)$  of the Clifford algebra of the gamma matrices  $\Gamma^a$ . According to the definition given in [1], these primitive elements are the representatives of the cohomology  $H_{\text{gh}}(s_{\text{gh}})$  of the coboundary operator

$$s_{\text{gh}} = \frac{i}{2} \delta^{ij} (\Gamma^a C^{-1})_{\underline{\alpha}\underline{\beta}} \xi_{\underline{\alpha}}^i \xi_{\underline{\beta}}^j \frac{\partial}{\partial c^a} \quad (1.5)$$

in the space  $\Omega_{\text{gh}}$  of polynomials in translation ghosts  $c^a$  and supersymmetry ghosts  $\xi_{\underline{\alpha}}^i$  corresponding to the translational generators  $P_a$  and the supersymmetry generators  $Q_{\underline{\alpha}}^i$  respectively,

$$\Omega_{\text{gh}} = \left\{ \sum_{p=0}^D \sum_{n=0}^r c^{a_1} \dots c^{a_p} \xi_{\underline{\alpha}_1}^{\alpha_1} \dots \xi_{\underline{\alpha}_n}^{\alpha_n} a_{\underline{\alpha}_1 \dots \underline{\alpha}_n a_1 \dots a_p}^{i_1 \dots i_n} \mid a_{\underline{\alpha}_1 \dots \underline{\alpha}_n a_1 \dots a_p}^{i_1 \dots i_n} \in \mathbb{C}, \quad r = 0, 1, 2, \dots \right\}. \quad (1.6)$$

Depending on the dimension  $D$  and on the signature  $(t, D - t)$  the supersymmetry generators and the supersymmetry ghosts are Majorana or symplectic Majorana spinors defined according to sections 2 and 4 of [1] by means of a matrix  $B$  and, in the case of symplectic Majorana spinors, a matrix  $\Omega$ :

$$\text{Majorana supersymmetries:} \quad \xi^{*i\bar{\alpha}} = \xi_{\underline{\alpha}}^{\beta} B^{-1}_{\beta}{}^{\bar{\alpha}}, \quad (1.7)$$

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<sup>1</sup>The index  $i = 1, \dots, N$  numbers sets of supersymmetries. In the case of Majorana-Weyl supersymmetries we use  $i = 1_+, \dots, N_+, 1_-, \dots, N_-$  with  $N = N_+ + N_-$  where the subscripts  $+$  and  $-$  indicate the chirality of the supersymmetries respectively.

$$\text{symplectic Majorana supersymmetries: } \xi^{*i\bar{\alpha}} = -\xi_j^\beta B^{-1}_{\beta}{}^{\bar{\alpha}} \Omega^{*ji} \quad (1.8)$$

where  $\xi^{*i\bar{\alpha}}$  denotes the conjugate-complex of  $\xi_i^\alpha$ . For the matrix  $B$  there are two different options in all cases except for signatures  $(0, 2)$  and  $(2, 0)$  in  $D = 2$  dimensions (see section 2.8 of [1]). The results will be formulated in such a way that they are valid for both choices of  $B$  (see section 5.5 of [1]). Therefore it will not be necessary to fix a choice of  $B$ . Symplectic Majorana supersymmetries occur only for the signatures  $(0, 3)$  and  $(3, 0)$  in  $D = 3$  dimensions and the matrix  $\Omega$  used in these cases will be specified in section 3.1.

In all cases we shall use the following strategy to compute  $H_{\text{gh}}(s_{\text{gh}})$ : we first compute the cohomology groups explicitly in a particular spinor representation and then reformulate the result in an  $\mathfrak{so}(t, D-t)$ -covariant way (with  $\mathfrak{so}(t, D-t)$ -transformations as in section 2.6 of [1]) so that they become independent of the spinor representation.

We shall use the notation  $\sim$  for equivalence in  $H_{\text{gh}}(s_{\text{gh}})$ , i.e. for  $\omega_1, \omega_2 \in \Omega_{\text{gh}}$  the notation  $\omega_1 \sim \omega_2$  means  $\omega_1 - \omega_2 = s_{\text{gh}}\omega_3$  for some  $\omega_3 \in \Omega_{\text{gh}}$ :

$$\omega_1 \sim \omega_2 \quad :\Leftrightarrow \quad \exists \omega_3 : \omega_1 - \omega_2 = s_{\text{gh}}\omega_3 \quad (\omega_1, \omega_2, \omega_3 \in \Omega_{\text{gh}}). \quad (1.9)$$

Notation and conventions which are not explained here are as in [1].

## 2 Primitive elements in $D = 2$ dimensions

### 2.1 $H_{\text{gh}}(s_{\text{gh}})$ for signature $(1, 1)$ in a particular representation

We shall first compute  $H_{\text{gh}}(s_{\text{gh}})$  in  $D = 2$  dimensions for signature  $(1, 1)$  and any numbers  $N_+, N_-$  of Majorana-Weyl supersymmetries in a spinor representation with

$$\Gamma^1 = -i\sigma_1, \quad \Gamma^2 = \sigma_2, \quad \hat{\Gamma} = \sigma_3, \quad C = \sigma_2. \quad (2.1)$$

In this spinor representation Majorana-Weyl supersymmetry ghosts  $\xi_{i\pm}^\pm = (\xi_{i\pm}^{\pm 1}, \xi_{i\pm}^{\pm 2})$  (with  $\xi_{i\pm}^\pm \hat{\Gamma} = \pm \xi_{i\pm}^\pm$ ) have only one nonvanishing component,

$$\xi_{i+}^+ = (\psi_{i+}, 0), \quad \xi_{i-}^- = (0, i\chi_{i-}). \quad (2.2)$$

The coboundary operator  $s_{\text{gh}}$  acts on the translation ghosts according to

$$s_{\text{gh}}c^1 = \frac{i}{2} \sum_{i_+=1}^{N_+} (\psi_{i_+})^2 + \frac{i}{2} \sum_{i_-=1}^{N_-} (\chi_{i_-})^2, \quad s_{\text{gh}}c^2 = \frac{i}{2} \sum_{i_+=1}^{N_+} (\psi_{i_+})^2 - \frac{i}{2} \sum_{i_-=1}^{N_-} (\chi_{i_-})^2. \quad (2.3)$$

These transformations can be simplified by introducing the following purely imaginary linear combinations of the translation ghosts:

$$\tilde{c}^1 = -i(c^1 + c^2), \quad \tilde{c}^2 = -i(c^1 - c^2). \quad (2.4)$$

$\tilde{c}^1$  and  $\tilde{c}^2$  have the  $s_{\text{gh}}$ -transformations

$$s_{\text{gh}}\tilde{c}^1 = \sum_{i_+=1}^{N_+} (\psi_{i_+})^2, \quad s_{\text{gh}}\tilde{c}^2 = \sum_{i_-=1}^{N_-} (\chi_{i_-})^2. \quad (2.5)$$

We define the space  $\Omega_+$  of polynomials  $\omega_+(\tilde{c}^1, \psi_{1_+}, \dots, \psi_{N_+})$  in  $\tilde{c}^1$  and the components of the supersymmetry ghosts of positive chirality, and the space  $\Omega_-$  of polynomials  $\omega_-(\tilde{c}^2, \chi_{1_-}, \dots, \chi_{N_-})$  in  $\tilde{c}^2$  and the components of the supersymmetry ghosts of negative chirality. Equation (2.5) shows that  $s_{\text{gh}}$  does not lead out of these spaces respectively, i.e.  $\omega_+ \in \Omega_+$  implies  $(s_{\text{gh}}\omega_+) \in \Omega_+$  and  $\omega_- \in \Omega_-$  implies  $(s_{\text{gh}}\omega_-) \in \Omega_-$ , for all  $\omega_+$  and  $\omega_-$ . Furthermore the space  $\Omega_{\text{gh}}$  of polynomials in all ghost variables can be written as the tensor product  $\Omega_+ \otimes \Omega_-$  of  $\Omega_+$  and  $\Omega_-$  (with  $\omega_+ \otimes \omega_- = \omega_+\omega_-$ ). This implies the Künneth formula  $H_{\text{gh}}(s_{\text{gh}}) = H_+(s_{\text{gh}}) \otimes H_-(s_{\text{gh}})$  where  $H_+(s_{\text{gh}})$  and  $H_-(s_{\text{gh}})$  denote the cohomology of  $s_{\text{gh}}$  in  $\Omega_+$  and  $\Omega_-$  respectively.

$H_+(s_{\text{gh}})$  and  $H_-(s_{\text{gh}})$  can be directly obtained from lemmas 6.1 and 6.2 of [1]. Indeed, up to a factor  $\frac{1}{2}$ ,  $s_{\text{gh}}\tilde{c}^1$  and  $s_{\text{gh}}\tilde{c}^2$  are completely analogous to  $s_{\text{gh}}c^1$  in  $D = 1$  dimension, cf. equation (6.5) of [1]. We conclude that  $H_+(s_{\text{gh}})$  is for  $N_+ > 0$  represented by polynomials  $a_0(\psi_{2_+}, \dots, \psi_{N_+}) + \psi_{1_+}a_1(\psi_{2_+}, \dots, \psi_{N_+})$  and that  $H_-(s_{\text{gh}})$  is for  $N_- > 0$  represented by polynomials  $b_0(\chi_{2_-}, \dots, \chi_{N_-}) + \chi_{1_-}b_1(\chi_{2_-}, \dots, \chi_{N_-})$ , where  $a_0(\psi_{2_+}, \dots, \psi_{N_+})$  and  $a_1(\psi_{2_+}, \dots, \psi_{N_+})$  are arbitrary polynomials in  $\psi_{2_+}, \dots, \psi_{N_+}$  and  $b_0(\chi_{2_-}, \dots, \chi_{N_-})$  and  $b_1(\chi_{2_-}, \dots, \chi_{N_-})$  are arbitrary polynomials in  $\chi_{2_-}, \dots, \chi_{N_-}$ . The Künneth formula yields thus:

**Lemma 2.1** ( $H_{\text{gh}}(s_{\text{gh}})$  for  $N_+ > 0$  and  $N_- > 0$ ).

*In the spinor representation (2.1),  $H_{\text{gh}}(s_{\text{gh}})$  is represented in the cases with both  $N_+ > 0$  and  $N_- > 0$  by polynomials in the supersymmetry ghosts which are at most linear both in  $\psi_{1_+}$  and in  $\chi_{1_-}$  and do not depend on the translation ghosts:*

$$s_{\text{gh}}\omega = 0 \Leftrightarrow \omega \sim a_{00} + \psi_{1_+}a_{10} + \chi_{1_-}a_{01} + \psi_{1_+}\chi_{1_-}a_{11}; \quad (2.6)$$

$$a_{00} + \psi_{1_+}a_{10} + \chi_{1_-}a_{01} + \psi_{1_+}\chi_{1_-}a_{11} \sim 0 \Leftrightarrow a_{00} = a_{10} = a_{01} = a_{11} = 0 \quad (2.7)$$

where  $a_{00}$ ,  $a_{10}$ ,  $a_{01}$  and  $a_{11}$  are polynomials in  $\psi_{2_+}, \dots, \psi_{N_+}$  or  $\chi_{2_-}, \dots, \chi_{N_-}$  or complex numbers:

$$N_+ > 1, N_- > 1: \quad a_{ij} = a_{ij}(\psi_{2_+}, \dots, \psi_{N_+}, \chi_{2_-}, \dots, \chi_{N_-}), \quad i, j \in \{0, 1\}; \quad (2.8)$$

$$N_+ > 1, N_- = 1: \quad a_{ij} = a_{ij}(\psi_{2_+}, \dots, \psi_{N_+}), \quad i, j \in \{0, 1\}; \quad (2.9)$$

$$N_+ = 1, N_- > 1: \quad a_{ij} = a_{ij}(\chi_{2_-}, \dots, \chi_{N_-}), \quad i, j \in \{0, 1\}; \quad (2.10)$$

$$N_+ = 1, N_- = 1: \quad a_{ij} \in \mathbb{C}, \quad i, j \in \{0, 1\}. \quad (2.11)$$

The cases  $N_+ = 0$  or  $N_- = 0$  are even simpler. E.g., in the case  $N_- = 0$  one has  $s_{\text{gh}}\tilde{c}^2 = 0$  and  $\Omega_- = \{a + b\tilde{c}^2 | a, b \in \mathbb{C}\}$ . Hence, in this case  $H_-(s_{\text{gh}})$  coincides with  $\Omega_-$  and the Künneth formula gives:

**Lemma 2.2** ( $H_{\text{gh}}(s_{\text{gh}})$  for  $N_+ > 0$  and  $N_- = 0$ ).

*In the spinor representation (2.1),  $H_{\text{gh}}(s_{\text{gh}})$  is represented in the cases with  $N_+ > 0$*

and  $N_- = 0$  by polynomials in the supersymmetry ghosts which are at most linear in  $\psi_{1+}$  and do not depend on  $\tilde{c}^1$ :

$$s_{\text{gh}}\omega = 0 \Leftrightarrow \omega \sim a_0 + \psi_{1+}a_1; \quad (2.12)$$

$$a_0 + \psi_{1+}a_1 \sim 0 \Leftrightarrow a_0 = a_1 = 0 \quad (2.13)$$

where  $a_0$  and  $a_1$  are polynomials in  $\psi_{2+}, \dots, \psi_{N+}$  or the translation ghost variable  $\tilde{c}^2$ :

$$N_+ > 1: \quad a_i = a_{i0}(\psi_{2+}, \dots, \psi_{N+}) + \tilde{c}^2 a_{i1}(\psi_{2+}, \dots, \psi_{N+}), \quad i \in \{0, 1\}; \quad (2.14)$$

$$N_+ = 1: \quad a_i = a_{i0} + \tilde{c}^2 a_{i1}, \quad a_{i0}, a_{i1} \in \mathbb{C}, \quad i \in \{0, 1\}. \quad (2.15)$$

An analogous result holds for  $N_+ = 0$  and  $N_- > 0$ , with the  $\chi_{i-}$  in place of the  $\psi_{i+}$  and  $\tilde{c}^1$  in place of  $\tilde{c}^2$ .

Notice that for two or more Majorana-Weyl supersymmetries it makes a considerable difference for the cohomology whether or not all the supersymmetries have the same chirality. In particular, in the case  $(N_+, N_-) = (1, 1)$  lemma 2.1 states that  $H_{\text{gh}}(s_{\text{gh}})$  is represented by  $a_{00} + \psi_{1+}a_{10} + \chi_{1-}a_{01} + \psi_{1+}\chi_{1-}a_{11}$  with  $a_{ij} \in \mathbb{C}$ . Hence,  $H_{\text{gh}}(s_{\text{gh}})$  is four dimensional in the case  $(N_+, N_-) = (1, 1)$  (counting complex dimensions). This differs from the case  $(N_+, N_-) = (2, 0)$  for which, according to lemma 2.2,  $H_{\text{gh}}(s_{\text{gh}})$  is represented by  $a_0(\psi_{2+}, \tilde{c}^2) + \psi_{1+}a_1(\psi_{2+}, \tilde{c}^2)$  where  $a_0(\psi_{2+}, \tilde{c}^2)$  and  $a_1(\psi_{2+}, \tilde{c}^2)$  are polynomials of arbitrary degree in  $\psi_{2+}$  and may also depend linearly on  $\tilde{c}^2$ . Hence, in the case  $(N_+, N_-) = (2, 0)$  the cohomology  $H_{\text{gh}}(s_{\text{gh}})$  is infinite dimensional, in sharp contrast to the case  $(N_+, N_-) = (1, 1)$  which has the same number of supersymmetries.

## 2.2 $H_{\text{gh}}(s_{\text{gh}})$ for signature (1,1) in covariant form

The results summarized in lemmas 2.1 and 2.2 can be readily rewritten for spinor representations equivalent to the spinor representation (2.1), using that the equivalence transformations relating any two spinor representations in even dimensions do not mix chiralities, cf. section 2.7 of [1]. Since the  $\psi_{i+}$  and  $\chi_{i-}$  denote the components of chiral supersymmetry ghosts in the spinor representation (2.1), we can simply substitute the components of chiral supersymmetry ghosts in any equivalent spinor representation for them to obtain  $H_{\text{gh}}(s_{\text{gh}})$  in the respective spinor representation. Furthermore, one readily checks that the product  $\psi_{1+}\chi_{1-}$  can be written as the  $\mathfrak{so}(1, 1)$ -invariant  $\xi_{1+}^{+\alpha}\xi_{1-}^{-\beta}(\hat{\Gamma}C^{-1})_{\alpha\beta}$  which extends it to spinor representations equivalent to (2.1). Therefore, a spinor representation independent formulation of lemma 2.1 is, for instance:

**Lemma 2.3** ( $H_{\text{gh}}(s_{\text{gh}})$  for  $N_+ > 0$  and  $N_- > 0$ ).

$H_{\text{gh}}(s_{\text{gh}})$  is represented in the cases with both  $N_+ > 0$  and  $N_- > 0$  by polynomials in the supersymmetry ghosts which are at most linear both in the components of  $\xi_{1+}^+$  and in the components of  $\xi_{1-}^-$  and do not depend on the translation ghosts:

$$s_{\text{gh}}\omega = 0 \Leftrightarrow \omega \sim a + \xi_{1+}^{+\alpha}a_{+\alpha} + \xi_{1-}^{-\alpha}a_{-\alpha} + \xi_{1+}^{+\alpha}\xi_{1-}^{-\beta}(\hat{\Gamma}C^{-1})_{\alpha\beta}a_{+-}; \quad (2.16)$$

$$\begin{aligned}
a + \xi_{1+}^{+\underline{\alpha}} a_{+\underline{\alpha}} + \xi_{1-}^{-\underline{\alpha}} a_{-\underline{\alpha}} + \xi_{1+}^{+\underline{\alpha}} \xi_{1-}^{-\underline{\beta}} (\hat{\Gamma} C^{-1})_{\underline{\alpha}\underline{\beta}} a_{+-} &\sim 0 \\
\Leftrightarrow a = \xi_{1+}^{+\underline{\alpha}} a_{+\underline{\alpha}} = \xi_{1-}^{-\underline{\alpha}} a_{-\underline{\alpha}} = a_{+-} &= 0
\end{aligned} \tag{2.17}$$

where  $a$ ,  $a_{+\underline{\alpha}}$ ,  $a_{-\underline{\alpha}}$  and  $a_{+-}$  are polynomials in the components of the supersymmetry ghosts  $\xi_{2+}^+, \dots, \xi_{N+}^+$  or  $\xi_{2-}^-, \dots, \xi_{N-}^-$  or complex numbers analogously to equations (2.8) to (2.11).

Analogously one may formulate lemma 2.2 in a spinor representation independent form. Equation (2.17) takes into account that, in general, in a spinor representation different from (but equivalent to) the spinor representation (2.1) the nonvanishing components of  $\xi_{1+}^+$  and of  $\xi_{1-}^-$  are linearly dependent, respectively.

### 2.3 $H_{\text{gh}}(s_{\text{gh}})$ for signatures (0,2) and (2,0)

We now derive  $H_{\text{gh}}(s_{\text{gh}})$  for signatures (0,2) and (2,0) departing from particular spinor representations with

$$\text{signature (0,2): } \Gamma^1 = \sigma_1, \Gamma^2 = \sigma_2, \hat{\Gamma} = \sigma_3, C = \sigma_2; \tag{2.18}$$

$$\text{signature (2,0): } \Gamma^1 = -i\sigma_1, \Gamma^2 = -i\sigma_2, \hat{\Gamma} = \sigma_3, C = \sigma_2. \tag{2.19}$$

$s_{\text{gh}}$  acts on the translation ghosts according to

$$\text{signature (0,2): } s_{\text{gh}} c^1 = \frac{1}{2} \sum_{i=1}^N (-\xi_i^1 \xi_i^1 + \xi_i^2 \xi_i^2), \quad s_{\text{gh}} c^2 = \frac{i}{2} \sum_{i=1}^N (\xi_i^1 \xi_i^1 + \xi_i^2 \xi_i^2); \tag{2.20}$$

$$\text{signature (2,0): } s_{\text{gh}} c^1 = \frac{i}{2} \sum_{i=1}^N (\xi_i^1 \xi_i^1 - \xi_i^2 \xi_i^2), \quad s_{\text{gh}} c^2 = \frac{1}{2} \sum_{i=1}^N (\xi_i^1 \xi_i^1 + \xi_i^2 \xi_i^2). \tag{2.21}$$

In terms of the ghost variables  $\psi_i = \xi_i^1$ ,  $\chi_i = \xi_i^2$  and

$$\text{signature (0,2): } \tilde{c}^1 = -c^1 - i c^2, \quad \tilde{c}^2 = c^1 - i c^2; \tag{2.22}$$

$$\text{signature (2,0): } \tilde{c}^1 = -i c^1 + c^2, \quad \tilde{c}^2 = i c^1 + c^2 \tag{2.23}$$

the  $s_{\text{gh}}$ -transformations (2.20) read in either case

$$s_{\text{gh}} \tilde{c}^1 = \sum_{i=1}^N (\psi_i)^2, \quad s_{\text{gh}} \tilde{c}^2 = \sum_{i=1}^N (\chi_i)^2. \tag{2.24}$$

These transformations are analogous to those in equation (2.5) for  $N_+ = N_- = N$ . Therefore we can directly obtain the cohomology  $H_{\text{gh}}(s_{\text{gh}})$  for signature (0,2) in the spinor representation (2.18) and for signature (2,0) in the spinor representation (2.19) from lemma 2.1 for  $N_+ = N_- = N$ :

**Lemma 2.4** ( $H_{\text{gh}}(s_{\text{gh}})$  in the particular spinor representations).

*In the spinor representations (2.18) for signature (0,2) and (2.19) for signature*

$(2, 0)$ ,  $H_{\text{gh}}(s_{\text{gh}})$  is represented by polynomials in the supersymmetry ghosts which are at most linear both in  $\psi_1$  and  $\chi_1$  and do not depend on the translation ghosts:

$$s_{\text{gh}}\omega = 0 \Leftrightarrow \omega \sim a_{00} + \psi_1 a_{10} + \chi_1 a_{01} + \psi_1 \chi_1 a_{11}; \quad (2.25)$$

$$a_{00} + \psi_1 a_{10} + \chi_1 a_{01} + \psi_1 \chi_1 a_{11} \sim 0 \Leftrightarrow a_{00} = a_{10} = a_{01} = a_{11} = 0 \quad (2.26)$$

where  $a_{00}$ ,  $a_{10}$ ,  $a_{01}$  and  $a_{11}$  are polynomials in  $\psi_2, \dots, \psi_N, \chi_2, \dots, \chi_N$  or complex numbers:

$$N > 1 : \quad a_{ij} = a_{ij}(\psi_2, \dots, \psi_N, \chi_2, \dots, \chi_N), \quad i, j \in \{0, 1\}; \quad (2.27)$$

$$N = 1 : \quad a_{ij} \in \mathbb{C}, \quad i, j \in \{0, 1\}. \quad (2.28)$$

To formulate  $H_{\text{gh}}(s_{\text{gh}})$  in spinor representations equivalent to the spinor representations (2.18) and (2.19) we use that  $\psi_1$  and  $\chi_1$  are the components of the supersymmetry ghost  $\xi_1$  and that the product  $\psi_1 \chi_1$  equals the  $\mathfrak{so}(t, 2-t)$  invariant  $\frac{i}{2} \xi_1^\alpha \xi_1^\beta (\hat{\Gamma} C^{-1})_{\alpha\beta}$  in these representations. This yields:

**Lemma 2.5** ( $H_{\text{gh}}(s_{\text{gh}})$  in covariant form).

$H_{\text{gh}}(s_{\text{gh}})$  is for signatures  $(0, 2)$  and  $(2, 0)$  represented by cocycles  $a$ ,  $\xi_1^\alpha a_{\underline{\alpha}}$  and  $\xi_1^\alpha \xi_1^\beta (\hat{\Gamma} C^{-1})_{\alpha\beta} a_{+-}$  where  $a$ ,  $a_{\underline{\alpha}}$  and  $a_{+-}$  are polynomials in the components of the supersymmetry ghosts  $\xi_2, \dots, \xi_N$  (if  $N > 1$ ) or complex numbers (if  $N = 1$ ) analogously to equations (2.27) and (2.28):

$$s_{\text{gh}}\omega = 0 \Leftrightarrow \omega \sim a + \xi_1^\alpha a_{\underline{\alpha}} + \xi_1^\alpha \xi_1^\beta (\hat{\Gamma} C^{-1})_{\alpha\beta} a_{+-}; \quad (2.29)$$

$$\begin{aligned} a + \xi_1^\alpha a_{\underline{\alpha}} + \xi_1^\alpha \xi_1^\beta (\hat{\Gamma} C^{-1})_{\alpha\beta} a_{+-} &\sim 0 \\ \Leftrightarrow \quad a = a_{\underline{\alpha}} = a_{+-} &= 0. \end{aligned} \quad (2.30)$$

### 3 Primitive elements in $D = 3$ dimensions

#### 3.1 $H_{\text{gh}}(s_{\text{gh}})$ in a particular representation

In  $D = 3$  dimensions we first compute  $H_{\text{gh}}(s_{\text{gh}})$  for any signature  $(t, 3-t)$  in a particular spinor representation given by

$$\Gamma_a = k_a \sigma_a, \quad a \in \{1, 2, 3\}, \quad k_a = \begin{cases} i & \text{for } a \leq t \\ 1 & \text{for } a > t \end{cases}, \quad C = \sigma_2. \quad (3.1)$$

The supersymmetry ghosts  $\xi_i$  are for signatures  $(1, 2)$  and  $(2, 1)$  Majorana spinors fulfilling equation (1.7) and for signatures  $(0, 3)$  and  $(3, 0)$  symplectic Majorana spinors fulfilling equation (1.8) with a matrix  $\Omega$  given by

$$\Omega = \begin{pmatrix} E & 0 & \cdots & 0 \\ 0 & E & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & E \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (3.2)$$

Accordingly, for signatures  $(0, 3)$  and  $(3, 0)$  we consider even numbers  $N$  of sets of supersymmetries and supersymmetry ghosts.

We introduce the following notation for the components of the supersymmetry ghosts:

$$(\xi_i^1, \xi_i^2) = (\psi_i, i\chi_i), \quad (3.3)$$

and the following translation ghost variables (with  $k_a$  as in (3.1)):

$$\tilde{c}^1 = -k_1 c^1 - i k_2 c^2, \quad \tilde{c}^2 = -k_1 c^1 + i k_2 c^2, \quad \tilde{c}^3 = -i k_3 c^3. \quad (3.4)$$

In terms of these ghost variables the coboundary operator  $s_{\text{gh}}$  acts for all signatures  $(t, 3-t)$  according to

$$s_{\text{gh}}\tilde{c}^1 = \sum_{i=1}^N (\psi_i)^2, \quad s_{\text{gh}}\tilde{c}^2 = \sum_{i=1}^N (\chi_i)^2, \quad s_{\text{gh}}\tilde{c}^3 = \sum_{i=1}^N \psi_i \chi_i. \quad (3.5)$$

### 3.1.1 Strategy

In order to compute  $H_{\text{gh}}(s_{\text{gh}})$  in the spinor representation (3.1) we shall use results in  $D = 2$  dimensions obtained in section 2. To use these results we define the space  $\hat{\Omega}$  of polynomials in the ghost variables that do not depend on  $\tilde{c}^3$ ,

$$\hat{\Omega} := \left\{ \hat{\omega} \in \Omega_{\text{gh}} \mid \frac{\partial \hat{\omega}}{\partial \tilde{c}^3} = 0 \right\}. \quad (3.6)$$

The coboundary operator  $s_{\text{gh}}$  acts in the space  $\hat{\Omega}$  exactly as on ghost polynomials in  $D = 2$  dimensions for signatures  $(0, 2)$  and  $(2, 0)$ , cf. equations (2.24). The cohomology of  $s_{\text{gh}}$  in  $\hat{\Omega}$  is thus obtained from lemma 2.4. We denote this cohomology by  $\hat{H}_{\text{gh}}(s_{\text{gh}})$ .

To determine  $H_{\text{gh}}(s_{\text{gh}})$  from  $\hat{H}_{\text{gh}}(s_{\text{gh}})$  we write a ghost polynomial  $\omega \in \Omega_{\text{gh}}$  as

$$\omega = \hat{\omega}_0 + \tilde{c}^3 \hat{\omega}_1, \quad \hat{\omega}_0, \hat{\omega}_1 \in \hat{\Omega}. \quad (3.7)$$

This yields

$$s_{\text{gh}}\omega = s_{\text{gh}}\hat{\omega}_0 + \sum_{i=1}^N \psi_i \chi_i \hat{\omega}_1 - \tilde{c}^3 (s_{\text{gh}}\hat{\omega}_1). \quad (3.8)$$

Notice that on the right hand side of equation (3.8) only the last term contains  $\tilde{c}^3$ . We thus obtain:

$$s_{\text{gh}}\omega = 0 \quad \Leftrightarrow \quad s_{\text{gh}}\hat{\omega}_1 = 0 \quad \wedge \quad s_{\text{gh}}\hat{\omega}_0 + \sum_{i=1}^N \psi_i \chi_i \hat{\omega}_1 = 0. \quad (3.9)$$



The first condition  $s_{\text{gh}}\hat{\omega}_1 = 0$  in (3.9) imposes that  $\hat{\omega}_1$  is a cocycle in  $\hat{H}_{\text{gh}}(s_{\text{gh}})$ . This condition will be solved by means of the result (2.25) of lemma 2.4. The second condition in (3.9) imposes that  $\sum_{i=1}^N \psi_i \chi_i \hat{\omega}_1$  is a coboundary in  $\hat{H}_{\text{gh}}(s_{\text{gh}})$ . That second condition will be solved by means of the result (2.26) of lemma 2.4. Then  $\hat{\omega}_0$  will be determined using again the result (2.25) of lemma 2.4, and  $\omega$  will be obtained from the results for  $\hat{\omega}_0$  and  $\hat{\omega}_1$  using (3.7). We shall have to distinguish the cases  $N = 1$  and  $N > 1$ .

### 3.1.2 $H_{\text{gh}}(s_{\text{gh}})$ for $N = 1$

Starting from the first condition  $s_{\text{gh}}\hat{\omega}_1 = 0$  in (3.9) we conclude in the case  $N = 1$  from the result (2.25) of lemma 2.4 that

$$\hat{\omega}_1 = s_{\text{gh}}\hat{\varrho}_1 + a_{00} + \psi_1 a_{10} + \chi_1 a_{01} + \psi_1 \chi_1 a_{11} \quad (3.10)$$

for some polynomial  $\hat{\varrho}_1 \in \hat{\Omega}$  and some complex numbers  $a_{ij} \in \mathbb{C}$ . Using this result for  $\hat{\omega}_1$  in the second condition in (3.9), the latter becomes in the case  $N = 1$ :

$$\begin{aligned} 0 &= s_{\text{gh}}\hat{\omega}_0 + \psi_1 \chi_1 (s_{\text{gh}}\hat{\varrho}_1 + a_{00} + \psi_1 a_{10} + \chi_1 a_{01} + \psi_1 \chi_1 a_{11}) \\ &= s_{\text{gh}}(\hat{\omega}_0 + \psi_1 \chi_1 \hat{\varrho}_1 + \tilde{c}^1 \chi_1 a_{10} + \tilde{c}^2 \psi_1 a_{01} + \frac{1}{2}(\tilde{c}^1 \chi_1 \chi_1 + \tilde{c}^2 \psi_1 \psi_1) a_{11}) + \psi_1 \chi_1 a_{00}. \end{aligned} \quad (3.11)$$

Equation (3.11) imposes in particular that  $\psi_1 \chi_1 a_{00}$  is a coboundary in  $\hat{H}_{\text{gh}}(s_{\text{gh}})$ . Using the result (2.26) of lemma 2.4 we conclude

$$a_{00} = 0. \quad (3.12)$$

Using now equation (3.12) in equation (3.11), the latter imposes

$$s_{\text{gh}}(\hat{\omega}_0 + \psi_1 \chi_1 \hat{\varrho}_1 + \tilde{c}^1 \chi_1 a_{10} + \tilde{c}^2 \psi_1 a_{01} + \frac{1}{2}(\tilde{c}^1 \chi_1 \chi_1 + \tilde{c}^2 \psi_1 \psi_1) a_{11}) = 0.$$

Using again the first result (2.25) of lemma 2.4, we conclude

$$\begin{aligned} \hat{\omega}_0 + \psi_1 \chi_1 \hat{\varrho}_1 + \tilde{c}^1 \chi_1 a_{10} + \tilde{c}^2 \psi_1 a_{01} + \frac{1}{2}(\tilde{c}^1 \chi_1 \chi_1 + \tilde{c}^2 \psi_1 \psi_1) a_{11} \\ = s_{\text{gh}}\hat{\varrho}_0 + b_{00} + \psi_1 b_{10} + \chi_1 b_{01} + \psi_1 \chi_1 b_{11} \end{aligned} \quad (3.13)$$

for some  $\hat{\varrho}_0 \in \hat{\Omega}$  and some  $b_{ij} \in \mathbb{C}$ . Solving equation (3.13) for  $\hat{\omega}_0$  and using the results for  $\hat{\omega}_0$  and  $\hat{\omega}_1$  in equation (3.7) we obtain

$$\begin{aligned} \omega &= -\psi_1 \chi_1 \hat{\varrho}_1 - \tilde{c}^1 \chi_1 a_{10} - \tilde{c}^2 \psi_1 a_{01} - \frac{1}{2}(\tilde{c}^1 \chi_1 \chi_1 + \tilde{c}^2 \psi_1 \psi_1) a_{11} \\ &\quad + s_{\text{gh}}\hat{\varrho}_0 + b_{00} + \psi_1 b_{10} + \chi_1 b_{01} + \psi_1 \chi_1 b_{11} \\ &\quad + \tilde{c}^3 (s_{\text{gh}}\hat{\varrho}_1 + \psi_1 a_{10} + \chi_1 a_{01} + \psi_1 \chi_1 a_{11}) \\ &= s_{\text{gh}}(\hat{\varrho}_0 - \tilde{c}^3 \hat{\varrho}_1 + \tilde{c}^3 b_{11}) + b_{00} + \psi_1 b_{10} + \chi_1 b_{01} + (\tilde{c}^3 \psi_1 - \tilde{c}^1 \chi_1) a_{10} \\ &\quad + (\tilde{c}^3 \chi_1 - \tilde{c}^2 \psi_1) a_{01} + (\tilde{c}^3 \psi_1 \chi_1 - \frac{1}{2}\tilde{c}^1 \chi_1 \chi_1 - \frac{1}{2}\tilde{c}^2 \psi_1 \psi_1) a_{11}. \end{aligned} \quad (3.14)$$

The cocycles  $b_{00} + \psi_1 b_{10} + \chi_1 b_{01} + (\tilde{c}^3 \psi_1 - \tilde{c}^1 \chi_1) a_{10} + (\tilde{c}^3 \chi_1 - \tilde{c}^2 \psi_1) a_{01}$  are at most linear in the supersymmetry ghosts and, therefore, cannot be exact in  $H_{\text{gh}}(s_{\text{gh}})$  since

$s_{\text{gh}}$ -coboundaries in  $\Omega_{\text{gh}}$  depend at least quadratically on the supersymmetry ghosts owing to equations (3.5). Furthermore it can be readily checked explicitly that  $\tilde{c}^3\psi_1\chi_1 - \frac{1}{2}\tilde{c}^1\chi_1\chi_1 - \frac{1}{2}\tilde{c}^2\psi_1\psi_1$  is not exact in  $H_{\text{gh}}(s_{\text{gh}})$ : in order to be a coboundary it would have to be of the form  $s_{\text{gh}}(d_{ab}c^ac^b)$  for some  $d_{ab} \in \mathbb{C}$  but no such  $d_{ab}$  exist. One can conclude the non-existence of the  $d_{ab}$  without any calculation, using that  $\tilde{c}^3\psi_1\chi_1 - \frac{1}{2}\tilde{c}^1\chi_1\chi_1 - \frac{1}{2}\tilde{c}^2\psi_1\psi_1$  is actually an  $\mathfrak{so}(t, 3-t)$ -invariant ghost polynomial, cf. section 3.2, and therefore, owing to the  $\mathfrak{so}(t, 3-t)$ -invariance of  $s_{\text{gh}}$ ,  $d_{ab}c^ac^b$  would have to be  $\mathfrak{so}(t, 3-t)$ -invariant too; however, there is no nonvanishing  $\mathfrak{so}(t, 3-t)$ -invariant bilinear polynomial in the translation ghosts in dimensions  $D \geq 3$  (the only candidate bilinear polynomial would be proportional to  $\eta_{ab}c^ac^b$  but this vanishes as the translation ghosts anticommute). We conclude:

**Lemma 3.1** ( $H_{\text{gh}}(s_{\text{gh}})$  for  $N = 1$ ).

*In the spinor representation (3.1),  $H_{\text{gh}}(s_{\text{gh}})$  is in the case  $N = 1$  represented by the cocycles  $1, \psi_1, \chi_1, \tilde{c}^3\psi_1 - \tilde{c}^1\chi_1, \tilde{c}^3\chi_1 - \tilde{c}^2\psi_1$  and  $\tilde{c}^3\psi_1\chi_1 - \frac{1}{2}\tilde{c}^1\chi_1\chi_1 - \frac{1}{2}\tilde{c}^2\psi_1\psi_1$ :*

$$s_{\text{gh}}\omega = 0 \Leftrightarrow \omega \sim b_{00} + \psi_1 b_{10} + \chi_1 b_{01} + (\tilde{c}^3\psi_1 - \tilde{c}^1\chi_1)a_{10} + (\tilde{c}^3\chi_1 - \tilde{c}^2\psi_1)a_{01} \\ + (\tilde{c}^3\psi_1\chi_1 - \frac{1}{2}\tilde{c}^1\chi_1\chi_1 - \frac{1}{2}\tilde{c}^2\psi_1\psi_1)a_{11}; \quad (3.15)$$

$$b_{00} + \psi_1 b_{10} + \chi_1 b_{01} + (\tilde{c}^3\psi_1 - \tilde{c}^1\chi_1)a_{10} + (\tilde{c}^3\chi_1 - \tilde{c}^2\psi_1)a_{01} \\ + (\tilde{c}^3\psi_1\chi_1 - \frac{1}{2}\tilde{c}^1\chi_1\chi_1 - \frac{1}{2}\tilde{c}^2\psi_1\psi_1)a_{11} \sim 0 \Leftrightarrow b_{ij} = a_{ij} = 0, \quad (3.16)$$

where  $b_{ij}, a_{ij} \in \mathbb{C}$ .

### 3.1.3 Towards $H_{\text{gh}}(s_{\text{gh}})$ for $N > 1$

In the cases  $N > 1$  we start again from the first condition  $s_{\text{gh}}\hat{\omega}_1 = 0$  in (3.9). We conclude from the result (2.25) for  $N > 1$  in lemma 2.4 that

$$\hat{\omega}_1 = s_{\text{gh}}\hat{\varrho}_1 + a_{00} + \psi_1 a_{10} + \chi_1 a_{01} + \psi_1 \chi_1 a_{11} \quad (3.17)$$

for some polynomial  $\hat{\varrho}_1 \in \hat{\Omega}$ , with  $a_{ij}$  polynomials in  $\psi_2, \dots, \psi_N, \chi_2, \dots, \chi_N$ :

$$a_{ij} = a_{ij}(\psi_2, \dots, \psi_N, \chi_2, \dots, \chi_N), \quad i, j \in \{0, 1\}. \quad (3.18)$$

Using this result for  $\hat{\omega}_1$  in the second condition in (3.9), the latter yields in the cases  $N > 1$ :

$$\begin{aligned} 0 &= s_{\text{gh}}\hat{\omega}_0 + \sum_i \psi_i \chi_i (s_{\text{gh}}\hat{\varrho}_1 + a_{00} + \psi_1 a_{10} + \chi_1 a_{01} + \psi_1 \chi_1 a_{11}) \\ &= s_{\text{gh}}(\hat{\omega}_0 + \sum_i \psi_i \chi_i \hat{\varrho}_1) + \psi_1 \chi_1 (a_{00} + \psi_1 a_{10} + \chi_1 a_{01} + \psi_1 \chi_1 a_{11}) \\ &\quad + \sum'_i \psi_i \chi_i (a_{00} + \psi_1 a_{10} + \chi_1 a_{01} + \psi_1 \chi_1 a_{11}) \\ &= s_{\text{gh}}(\hat{\omega}_0 + \sum_i \psi_i \chi_i \hat{\varrho}_1 + \tilde{c}^1\chi_1 a_{10} + \tilde{c}^2\psi_1 a_{01} + \frac{1}{2}(\tilde{c}^1\chi_1\chi_1 + \tilde{c}^2\psi_1\psi_1)a_{11}) \\ &\quad + \psi_1 \chi_1 a_{00} - \sum'_i (\psi_i \psi_i \chi_1 a_{10} + \chi_i \chi_i \psi_1 a_{01}) - \frac{1}{2} \sum'_i (\psi_i \psi_i \chi_1 \chi_1 + \chi_i \chi_i \psi_1 \psi_1) a_{11} \\ &\quad + \sum'_i \psi_i \chi_i (a_{00} + \psi_1 a_{10} + \chi_1 a_{01} + \psi_1 \chi_1 a_{11}) \\ &= s_{\text{gh}}(\hat{\omega}_0 + \sum_i \psi_i \chi_i \hat{\varrho}_1 + \tilde{c}^1\chi_1 a_{10} + \tilde{c}^2\psi_1 a_{01} \\ &\quad + \frac{1}{2}(\tilde{c}^1\chi_1\chi_1 + \tilde{c}^2\psi_1\psi_1 - \sum'_i \tilde{c}^2\psi_i \psi_i - \sum'_i \tilde{c}^1\chi_i \chi_i) a_{11}) \end{aligned}$$

$$\begin{aligned}
& + \psi_1 \chi_1 a_{00} - \Sigma'_i (\psi_i \psi_i \chi_1 a_{10} + \chi_i \chi_i \psi_1 a_{01}) + \Sigma'_i \psi_i \psi_i \Sigma'_j \chi_j \chi_j a_{11} \\
& + \Sigma'_i \psi_i \chi_i (a_{00} + \psi_1 a_{10} + \chi_1 a_{01} + \psi_1 \chi_1 a_{11})
\end{aligned} \tag{3.19}$$

where we used the notation

$$\Sigma_i := \sum_{i=1}^N, \quad \Sigma'_i := \sum_{i=2}^N.$$

Equation (3.19) imposes:

$$\begin{aligned}
s_{\text{gh}} \hat{\rho}_2 &= \psi_1 \chi_1 (a_{00} + \Sigma'_i \psi_i \chi_i a_{11}) + \Sigma'_i \psi_i \chi_i a_{00} + \Sigma'_i \psi_i \psi_i \Sigma'_j \chi_j \chi_j a_{11} \\
&+ \psi_1 \Sigma'_i (\psi_i \chi_i a_{10} - \chi_i \chi_i a_{01}) + \chi_1 \Sigma'_i (\psi_i \chi_i a_{01} - \psi_i \psi_i a_{10})
\end{aligned} \tag{3.20}$$

for  $\hat{\rho}_2 = (-\hat{\omega}_0 + \dots) \in \hat{\Omega}$ . Using the result (2.26) we conclude from equation (3.20) that

$$\begin{aligned}
a_{00} + \Sigma'_i \psi_i \chi_i a_{11} &= 0, \quad \Sigma'_i \psi_i \chi_i a_{00} + \Sigma'_i \psi_i \psi_i \Sigma'_j \chi_j \chi_j a_{11} = 0, \\
\Sigma'_i (\psi_i \chi_i a_{10} - \chi_i \chi_i a_{01}) &= 0, \quad \Sigma'_i (\psi_i \chi_i a_{01} - \psi_i \psi_i a_{10}) = 0.
\end{aligned} \tag{3.21}$$

The first and the second of these conditions imply

$$(\Sigma'_i \psi_i \chi_i \Sigma'_j \psi_j \chi_j - \Sigma'_i \psi_i \psi_i \Sigma'_j \chi_j \chi_j) a_{11} = 0 \tag{3.22}$$

which holds identically (for any  $a_{11}$ ) in the case  $N = 2$  and imposes  $a_{11} = 0$  in the cases  $N > 2$ .

The third and the fourth of the conditions in (3.21) give:

$$N = 2 : \quad \psi_2 a_{10} - \chi_2 a_{01} = 0; \tag{3.23}$$

$$\begin{aligned}
N > 2 : \quad a_{10} (\Sigma'_i \psi_i \chi_i \Sigma'_j \psi_j \chi_j - \Sigma'_i \psi_i \psi_i \Sigma'_j \chi_j \chi_j) &= 0, \\
a_{01} (\Sigma'_i \psi_i \chi_i \Sigma'_j \psi_j \chi_j - \Sigma'_i \psi_i \psi_i \Sigma'_j \chi_j \chi_j) &= 0.
\end{aligned} \tag{3.24}$$

(3.23) and (3.24) imply

$$N = 2 : \quad a_{10} = \chi_2 b, \quad a_{01} = \psi_2 b; \tag{3.25}$$

$$N > 2 : \quad a_{10} = a_{01} = 0 \tag{3.26}$$

for some polynomial  $b$  in  $\psi_2$  and  $\chi_2$ .

We thus infer from (3.21):

$$N = 2 : \quad a_{00} = -\psi_2 \chi_2 a_{11}, \quad a_{10} = \chi_2 b, \quad a_{01} = \psi_2 b; \tag{3.27}$$

$$N > 2 : \quad a_{00} = a_{11} = a_{10} = a_{01} = 0 \tag{3.28}$$

where in (3.27)  $a_{11} = a_{11}(\psi_2, \chi_2)$  and  $b = b(\psi_2, \chi_2)$  are polynomials in  $\psi_2$  and  $\chi_2$  which are not constrained by the cocycle condition. To proceed, we have to distinguish the cases  $N = 2$  and  $N > 2$ .

### 3.1.4 $H_{\text{gh}}(s_{\text{gh}})$ for $N = 2$

Using (3.27) in equations (3.17) and (3.19) we obtain

$$\hat{\omega}_1 = s_{\text{gh}}\hat{\varrho}_1 + (\psi_1\chi_2 + \chi_1\psi_2)b + (\psi_1\chi_1 - \psi_2\chi_2)a_{11}; \quad (3.29)$$

$$\begin{aligned} s_{\text{gh}}(\hat{\omega}_0 + \Sigma_i \psi_i \chi_i \hat{\varrho}_1 + (\tilde{c}^1 \chi_1 \chi_2 + \tilde{c}^2 \psi_1 \psi_2)b \\ + \frac{1}{2}(\tilde{c}^1 \chi_1 \chi_1 + \tilde{c}^2 \psi_1 \psi_1 - \tilde{c}^2 \psi_2 \psi_2 - \tilde{c}^1 \chi_2 \chi_2)a_{11}) = 0. \end{aligned} \quad (3.30)$$

Using once again the result (2.25) of lemma 2.4 we infer from (3.30):

$$\begin{aligned} \hat{\omega}_0 + \Sigma_i \psi_i \chi_i \hat{\varrho}_1 + (\tilde{c}^1 \chi_1 \chi_2 + \tilde{c}^2 \psi_1 \psi_2)b \\ + \frac{1}{2}(\tilde{c}^1(\chi_1 \chi_1 - \chi_2 \chi_2) + \tilde{c}^2(\psi_1 \psi_1 - \psi_2 \psi_2))a_{11} \\ = s_{\text{gh}}\hat{\varrho}_0 + b_{00} + \psi_1 b_{10} + \chi_1 b_{01} + \psi_1 \chi_1 b_{11} \end{aligned} \quad (3.31)$$

for some  $\hat{\varrho}_0 \in \hat{\Omega}$  and some polynomials  $b_{ij} = b_{ij}(\psi_2, \chi_2)$  in  $\psi_2$  and  $\chi_2$ . Using the results for  $\hat{\omega}_0$  and  $\hat{\omega}_1$  in equation (3.7), we obtain:

$$\begin{aligned} \omega = & -\Sigma_i \psi_i \chi_i \hat{\varrho}_1 - (\tilde{c}^1 \chi_1 \chi_2 + \tilde{c}^2 \psi_1 \psi_2)b \\ & - \frac{1}{2}(\tilde{c}^1(\chi_1 \chi_1 - \chi_2 \chi_2) + \tilde{c}^2(\psi_1 \psi_1 - \psi_2 \psi_2))a_{11} \\ & + s_{\text{gh}}\hat{\varrho}_0 + b_{00} + \psi_1 b_{10} + \chi_1 b_{01} + \psi_1 \chi_1 b_{11} \\ & + \tilde{c}^3(s_{\text{gh}}\hat{\varrho}_1 + (\psi_1 \chi_2 + \chi_1 \psi_2)b + (\psi_1 \chi_1 - \psi_2 \chi_2)a_{11}) \\ = & s_{\text{gh}}(\hat{\varrho}_0 - \tilde{c}^3 \hat{\varrho}_1 + \tilde{c}^3 b_{11}) + b'_{00} + \psi_1 b_{10} + \chi_1 b_{01} \\ & + (\tilde{c}^3(\psi_1 \chi_2 + \chi_1 \psi_2) - \tilde{c}^1 \chi_1 \chi_2 - \tilde{c}^2 \psi_1 \psi_2)b \\ & + (\tilde{c}^3(\psi_1 \chi_1 - \psi_2 \chi_2) - \frac{1}{2}\tilde{c}^1(\chi_1 \chi_1 - \chi_2 \chi_2) - \frac{1}{2}\tilde{c}^2(\psi_1 \psi_1 - \psi_2 \psi_2))a_{11} \end{aligned} \quad (3.32)$$

where  $b'_{00} = b_{00} - \psi_2 \chi_2 b_{11}$ .

We have thus shown:

$$\begin{aligned} s_{\text{gh}}\omega = 0 \Leftrightarrow \omega \sim & b'_{00} + \psi_1 b_{10} + \chi_1 b_{01} + (\tilde{c}^3(\psi_1 \chi_2 + \chi_1 \psi_2) - \tilde{c}^1 \chi_1 \chi_2 - \tilde{c}^2 \psi_1 \psi_2)b \\ & + (\tilde{c}^3(\psi_1 \chi_1 - \psi_2 \chi_2) - \frac{1}{2}\tilde{c}^1(\chi_1 \chi_1 - \chi_2 \chi_2) - \frac{1}{2}\tilde{c}^2(\psi_1 \psi_1 - \psi_2 \psi_2))a_{11}. \end{aligned} \quad (3.33)$$

We shall now investigate whether and which cocycles in equation (3.33) are coboundaries, i.e., we shall study the equation

$$\begin{aligned} b'_{00} + \psi_1 b_{10} + \chi_1 b_{01} + (\tilde{c}^3(\psi_1 \chi_2 + \chi_1 \psi_2) - \tilde{c}^1 \chi_1 \chi_2 - \tilde{c}^2 \psi_1 \psi_2)b \\ + (\tilde{c}^3(\psi_1 \chi_1 - \psi_2 \chi_2) - \frac{1}{2}\tilde{c}^1(\chi_1 \chi_1 - \chi_2 \chi_2) - \frac{1}{2}\tilde{c}^2(\psi_1 \psi_1 - \psi_2 \psi_2))a_{11} = s_{\text{gh}}\varrho. \end{aligned} \quad (3.34)$$

As in equation (3.7) we write  $\varrho$  as

$$\varrho = \hat{\varrho}_3 + \tilde{c}^3 \hat{\varrho}_4, \quad \hat{\varrho}_3, \hat{\varrho}_4 \in \hat{\Omega}. \quad (3.35)$$

Writing  $s_{\text{gh}}\varrho$  analogously to  $s_{\text{gh}}\omega$  in (3.8), equation (3.34) becomes

$$b'_{00} + \psi_1 b_{10} + \chi_1 b_{01} + (\tilde{c}^3(\psi_1 \chi_2 + \chi_1 \psi_2) - \tilde{c}^1 \chi_1 \chi_2 - \tilde{c}^2 \psi_1 \psi_2)b$$

$$\begin{aligned}
& + (\tilde{c}^3(\psi_1\chi_1 - \psi_2\chi_2) - \frac{1}{2}\tilde{c}^1(\chi_1\chi_1 - \chi_2\chi_2) - \frac{1}{2}\tilde{c}^2(\psi_1\psi_1 - \psi_2\psi_2))a_{11} \\
& = s_{\text{gh}}\hat{\varrho}_3 + (\psi_1\chi_1 + \psi_2\chi_2)\hat{\varrho}_4 - \tilde{c}^3(s_{\text{gh}}\hat{\varrho}_4).
\end{aligned} \tag{3.36}$$

The terms in (3.36) containing  $\tilde{c}^3$  impose

$$(\psi_1\chi_2 + \chi_1\psi_2)b + (\psi_1\chi_1 - \psi_2\chi_2)a_{11} = -s_{\text{gh}}\hat{\varrho}_4. \tag{3.37}$$

Using the result (2.26) of lemma 2.4, we conclude from equation (3.37)

$$b = a_{11} = 0; \tag{3.38}$$

$$s_{\text{gh}}\hat{\varrho}_4 = 0. \tag{3.39}$$

Using the result (2.25) of lemma 2.4, we infer from (3.39):

$$\hat{\varrho}_4 = s_{\text{gh}}\hat{\varrho}' + d_{00} + \psi_1d_{10} + \chi_1d_{01} + \psi_1\chi_1d_{11}, \quad \hat{\varrho}' \in \hat{\Omega} \tag{3.40}$$

with  $d_{ij} = d_{ij}(\psi_2, \chi_2)$  polynomials in  $\psi_2$  and  $\chi_2$ .

Using now (3.38) and (3.40) in equation (3.36), the latter yields

$$\begin{aligned}
b'_{00} + \psi_1b_{10} + \chi_1b_{01} &= s_{\text{gh}}(\hat{\varrho}_3 + (\psi_1\chi_1 + \psi_2\chi_2)\hat{\varrho}') \\
&+ (\psi_1\chi_1 + \psi_2\chi_2)(d_{00} + \psi_1d_{10} + \chi_1d_{01} + \psi_1\chi_1d_{11}) \\
&= s_{\text{gh}}(\hat{\varrho}_3 + (\psi_1\chi_1 + \psi_2\chi_2)\hat{\varrho}') \\
&+ (\psi_1\chi_1 + \psi_2\chi_2)d_{00} \\
&+ s_{\text{gh}}(\tilde{c}^1\chi_1d_{10} - \psi_2\psi_2\chi_1d_{10} + \psi_2\chi_2\psi_1d_{10} \\
&+ s_{\text{gh}}(\tilde{c}^2\psi_1d_{01} - \chi_2\chi_2\psi_1d_{01} + \psi_2\chi_2\chi_1d_{01} \\
&+ s_{\text{gh}}(\tilde{c}^1\chi_1\chi_1d_{11} - \tilde{c}^2\psi_2\psi_2d_{11}) \\
&+ \psi_2\psi_2\chi_2\chi_2d_{11} + \psi_2\chi_2\psi_1\chi_1d_{11}.
\end{aligned} \tag{3.41}$$

This gives:

$$\begin{aligned}
& b'_{00} - \psi_2\chi_2d_{00} - \psi_2\psi_2\chi_2\chi_2d_{11} + \psi_1(b_{10} + \chi_2\chi_2d_{01} - \psi_2\chi_2d_{10}) \\
& + \chi_1(b_{01} + \psi_2\psi_2d_{10} - \psi_2\chi_2d_{01}) - \psi_1\chi_1(d_{00} + \psi_2\chi_2d_{11}) = s_{\text{gh}}\hat{\varrho}_5
\end{aligned} \tag{3.42}$$

for  $\hat{\varrho}_5 = (\hat{\varrho}_3 + \dots) \in \hat{\Omega}$ . Using the result (2.26) of lemma 2.4 we infer from (3.42):

$$b'_{00} - \psi_2\chi_2d_{00} - \psi_2\psi_2\chi_2\chi_2d_{11} = 0, \quad d_{00} + \psi_2\chi_2d_{11} = 0, \tag{3.43}$$

$$b_{10} + \chi_2\chi_2d_{01} - \psi_2\chi_2d_{10} = 0, \quad b_{01} + \psi_2\psi_2d_{10} - \psi_2\chi_2d_{01} = 0. \tag{3.44}$$

(3.43) implies  $b'_{00} = 0$ , (3.44) provides those polynomials  $b_{10}$ ,  $b_{01}$  for which  $\psi_1b_{10} + \chi_1b_{01}$  is a coboundary in  $H_{\text{gh}}(s_{\text{gh}})$ . The latter condition on  $b_{10}$  and  $b_{01}$  can be rewritten in terms of the cocycle  $\psi_1b_{10} + \chi_1b_{01}$  as follows:

$$\begin{aligned}
\psi_1b_{10} + \chi_1b_{01} &= (\psi_1\psi_2\chi_2 - \chi_1\psi_2\psi_2)d_{10} + (\chi_1\psi_2\chi_2 - \psi_1\chi_2\chi_2)d_{01} \\
&= (\psi_1\chi_2 - \chi_1\psi_2)(\psi_2d_{10} - \chi_2d_{01}).
\end{aligned} \tag{3.45}$$

We have thus shown:

**Lemma 3.2** ( $H_{\text{gh}}(s_{\text{gh}})$  for  $N = 2$ ).

In the spinor representation (3.1),  $H_{\text{gh}}(s_{\text{gh}})$  is in the case  $N = 2$  represented by cocycles  $b'_{00}$ ,  $\psi_1 b_{10}$ ,  $\chi_1 b_{01}$ ,  $(\tilde{c}^3(\psi_1 \chi_2 + \chi_1 \psi_2) - \tilde{c}^1 \chi_1 \chi_2 - \tilde{c}^2 \psi_1 \psi_2)b$  and  $(\tilde{c}^3(\psi_1 \chi_1 - \psi_2 \chi_2) - \frac{1}{2}\tilde{c}^1(\chi_1 \chi_1 - \chi_2 \chi_2) - \frac{1}{2}\tilde{c}^2(\psi_1 \psi_1 - \psi_2 \psi_2))a_{11}$  where  $b'_{00}$ ,  $b_{10}$ ,  $b_{01}$ ,  $b$  and  $a_{11}$  are polynomials in  $\psi_2$  and  $\chi_2$ :

$$s_{\text{gh}}\omega = 0 \Leftrightarrow \omega \sim b'_{00} + \psi_1 b_{10} + \chi_1 b_{01} + (\tilde{c}^3(\psi_1 \chi_2 + \chi_1 \psi_2) - \tilde{c}^1 \chi_1 \chi_2 - \tilde{c}^2 \psi_1 \psi_2)b + (\tilde{c}^3(\psi_1 \chi_1 - \psi_2 \chi_2) - \frac{1}{2}\tilde{c}^1(\chi_1 \chi_1 - \chi_2 \chi_2) - \frac{1}{2}\tilde{c}^2(\psi_1 \psi_1 - \psi_2 \psi_2))a_{11}; \quad (3.46)$$

$$\begin{aligned} & b'_{00} + \psi_1 b_{10} + \chi_1 b_{01} + (\tilde{c}^3(\psi_1 \chi_2 + \chi_1 \psi_2) - \tilde{c}^1 \chi_1 \chi_2 - \tilde{c}^2 \psi_1 \psi_2)b \\ & + (\tilde{c}^3(\psi_1 \chi_1 - \psi_2 \chi_2) - \frac{1}{2}\tilde{c}^1(\chi_1 \chi_1 - \chi_2 \chi_2) - \frac{1}{2}\tilde{c}^2(\psi_1 \psi_1 - \psi_2 \psi_2))a_{11} \sim 0 \\ \Leftrightarrow & b'_{00} = b = a_{11} = 0 \wedge \psi_1 b_{10} + \chi_1 b_{01} = (\psi_1 \chi_2 - \chi_1 \psi_2)(\psi_2 d_{10} - \chi_2 d_{01}) \end{aligned} \quad (3.47)$$

for some polynomials  $d_{10}$  and  $d_{01}$  in  $\psi_2$  and  $\chi_2$ .

### 3.1.5 $H_{\text{gh}}(s_{\text{gh}})$ for $N > 2$

Using (3.28) in equations (3.17) and (3.19) we obtain

$$\hat{\omega}_1 = s_{\text{gh}}\hat{\varrho}_1; \quad (3.48)$$

$$s_{\text{gh}}(\hat{\omega}_0 + \Sigma_i \psi_i \chi_i \hat{\varrho}_1) = 0. \quad (3.49)$$

Using once again the result (2.25) of lemma 2.4 we conclude from (3.49):

$$\hat{\omega}_0 + \Sigma_i \psi_i \chi_i \hat{\varrho}_1 = s_{\text{gh}}\hat{\varrho}_0 + b_{00} + \psi_1 b_{10} + \chi_1 b_{01} + \psi_1 \chi_1 b_{11} \quad (3.50)$$

for some  $\hat{\varrho}_0 \in \hat{\Omega}$  and some polynomials  $b_{ij} = b_{ij}(\psi_2, \dots, \psi_N, \chi_2, \dots, \chi_N)$ . Using the results for  $\hat{\omega}_0$  and  $\hat{\omega}_1$  in equation (3.7), we obtain:

$$\begin{aligned} \omega &= -\Sigma_i \psi_i \chi_i \hat{\varrho}_1 + s_{\text{gh}}\hat{\varrho}_0 + b_{00} + \psi_1 b_{10} + \chi_1 b_{01} + \psi_1 \chi_1 b_{11} + \tilde{c}^3(s_{\text{gh}}\hat{\varrho}_1) \\ &= s_{\text{gh}}(\hat{\varrho}_0 - \tilde{c}^3\hat{\varrho}_1 + \tilde{c}^3 b_{11}) + b'_{00} + \psi_1 b_{10} + \chi_1 b_{01} \end{aligned} \quad (3.51)$$

with  $b'_{00} = b_{00} - \Sigma'_i \psi_i \psi_i b_{11}$  with  $\Sigma'_i = \Sigma_{i=2}^N$ . We have thus shown in the cases  $N > 2$ :

$$s_{\text{gh}}\omega = 0 \Leftrightarrow \omega \sim b'_{00} + \psi_1 b_{10} + \chi_1 b_{01}. \quad (3.52)$$

We still have to determine those cocycles  $b'_{00} + \psi_1 b_{10} + \chi_1 b_{01}$  that are coboundaries. Analogously to the case  $N = 2$  we thus study the equation

$$b'_{00} + \psi_1 b_{10} + \chi_1 b_{01} = s_{\text{gh}}\varrho. \quad (3.53)$$

In the same way as in the analysis of equation (3.34) one derives the analog of (3.41) for  $N > 2$ :

$$\begin{aligned} b'_{00} + \psi_1 b_{10} + \chi_1 b_{01} &= s_{\text{gh}}(\hat{\varrho}_3 + \Sigma_i \psi_i \chi_i \hat{\varrho}') \\ &+ \Sigma_i \psi_i \chi_i (d_{00} + \psi_1 d_{10} + \chi_1 d_{01} + \psi_1 \chi_1 d_{11}) \end{aligned}$$

$$\begin{aligned}
&= s_{\text{gh}}(\hat{\varrho}_3 + \Sigma_i \psi_i \chi_i \hat{\varrho}') \\
&\quad + \Sigma_i \psi_i \chi_i d_{00} \\
&\quad + s_{\text{gh}}(\tilde{c}^1 \chi_1 d_{10}) - \Sigma'_i \psi_i \psi_i \chi_1 d_{10} + \Sigma'_i \psi_i \chi_i \psi_1 d_{10} \\
&\quad + s_{\text{gh}}(\tilde{c}^2 \psi_1 d_{01}) - \Sigma'_i \chi_i \chi_i \psi_1 d_{01} + \Sigma'_i \psi_i \chi_i \chi_1 d_{01} \\
&\quad + s_{\text{gh}}(\tilde{c}^1 \chi_1 \chi_1 d_{11} - \tilde{c}^2 \Sigma'_i \psi_i \psi_i d_{11}) \\
&\quad + \Sigma'_i \psi_i \psi_i \Sigma'_j \chi_j \chi_j d_{11} + \Sigma'_i \psi_i \chi_i \psi_1 \chi_1 d_{11} .
\end{aligned} \tag{3.54}$$

This yields:

$$\begin{aligned}
s_{\text{gh}} \hat{\varrho}_5 &= b'_{00} - \Sigma'_i \psi_i \chi_i d_{00} - \Sigma'_i \psi_i \psi_i \Sigma'_j \chi_j \chi_j d_{11} \\
&\quad + \psi_1 (b_{10} + \Sigma'_i \chi_i \chi_i d_{01} - \Sigma'_i \psi_i \chi_i d_{10}) \\
&\quad + \chi_1 (b_{01} + \Sigma'_i \psi_i \psi_i d_{10} - \Sigma'_i \psi_i \chi_i d_{01}) \\
&\quad - \psi_1 \chi_1 (d_{00} + \Sigma'_i \psi_i \chi_i d_{11})
\end{aligned} \tag{3.55}$$

with  $\hat{\varrho}_5 = (\hat{\varrho}_3 + \dots) \in \hat{\Omega}$ . Using the result (2.26) of lemma 2.4 we infer from (3.55):

$$b'_{00} = \Sigma'_i \psi_i \chi_i d_{00} + \Sigma'_i \psi_i \psi_i \Sigma'_j \chi_j \chi_j d_{11}, \quad d_{00} + \Sigma'_i \psi_i \chi_i d_{11} = 0, \tag{3.56}$$

$$b_{10} = -\Sigma'_i \chi_i \chi_i d_{01} + \Sigma'_i \psi_i \chi_i d_{10}, \quad b_{01} = -\Sigma'_i \psi_i \psi_i d_{10} + \Sigma'_i \psi_i \chi_i d_{01}. \tag{3.57}$$

(3.56) implies

$$\begin{aligned}
b'_{00} &= (-\Sigma'_i \psi_i \chi_i \Sigma'_j \psi_j \chi_j + \Sigma'_i \psi_i \psi_i \Sigma'_j \chi_j \chi_j) d_{11} \\
&= \frac{1}{2} \Sigma'_i \Sigma'_j (\psi_i \chi_j - \chi_i \psi_j)^2 d_{11} .
\end{aligned} \tag{3.58}$$

(3.57) implies

$$\begin{aligned}
\psi_1 b_{10} + \chi_1 b_{01} &= \psi_1 (-\Sigma'_i \chi_i \chi_i d_{01} + \Sigma'_i \psi_i \chi_i d_{10}) + \chi_1 (-\Sigma'_i \psi_i \psi_i d_{10} + \Sigma'_i \psi_i \chi_i d_{01}) \\
&= \Sigma'_i \psi_i (\psi_1 \chi_i - \chi_1 \psi_i) d_{10} - \Sigma'_i \chi_i (\psi_1 \chi_i - \chi_1 \psi_i) d_{01} .
\end{aligned} \tag{3.59}$$

This yields:

**Lemma 3.3** ( $H_{\text{gh}}(s_{\text{gh}})$  for  $N > 2$ ).

In the spinor representation (3.1),  $H_{\text{gh}}(s_{\text{gh}})$  is in the case  $N > 2$  represented by cocycles  $b'_{00}$ ,  $\psi_1 b_{10}$  and  $\chi_1 b_{01}$  where  $b'_{00}$ ,  $b_{10}$  and  $b_{01}$  are polynomials in  $\psi_2, \dots, \psi_N, \chi_2, \dots, \chi_N$ :

$$s_{\text{gh}} \omega = 0 \Leftrightarrow \omega \sim b'_{00} + \psi_1 b_{10} + \chi_1 b_{01}; \tag{3.60}$$

$$b'_{00} + \psi_1 b_{10} + \chi_1 b_{01} \sim 0 \Leftrightarrow$$

$$\begin{aligned}
b'_{00} &= \frac{1}{2} \sum_{i=2}^N \sum_{j=2}^N (\psi_i \chi_j - \chi_i \psi_j)^2 d_{11} \wedge \\
\psi_1 b_{10} + \chi_1 b_{01} &= \sum_{i=2}^N [\psi_i (\psi_1 \chi_i - \chi_1 \psi_i) d_{10} - \chi_i (\psi_1 \chi_i - \chi_1 \psi_i) d_{01}]
\end{aligned} \tag{3.61}$$

for some polynomials  $d_{11}$ ,  $d_{10}$  and  $d_{01}$  in  $\psi_2, \dots, \psi_N, \chi_2, \dots, \chi_N$ .

### 3.2 $H_{\text{gh}}(s_{\text{gh}})$ in covariant form

To derive a spinor representation independent formulation of lemmas 3.1, 3.2 and 3.3 we introduce the following  $\mathfrak{so}(t, 3-t)$ -covariant ghost polynomials:

$$\vartheta_i^\alpha = c^a \xi_i^\beta \Gamma_{a\beta}^\alpha, \quad \Theta_{ij} = c^a \xi_i^\alpha \xi_j^\beta (\Gamma_a C^{-1})_{\alpha\beta}. \quad (3.62)$$

The  $\Theta_{ij}$  fulfill:

$$\Theta_{ij} = \Theta_{ji} = \vartheta_i \cdot \xi_j = \vartheta_j \cdot \xi_i \quad (3.63)$$

where  $\vartheta_i \cdot \xi_j$  denotes the  $\mathfrak{so}(t, 3-t)$ -invariant product of  $\vartheta_i$  and  $\xi_j$ , cf. equation (2.39) of [1].

In the spinor representation (3.1) one has, for all signatures  $(t, 3-t)$ ,

$$\vartheta_i^1 = i(\tilde{c}^3 \psi_i - \tilde{c}^1 \chi_i), \quad \vartheta_i^2 = \tilde{c}^3 \chi_i - \tilde{c}^2 \psi_i, \quad (3.64)$$

$$\Theta_{ij} = i(\tilde{c}^3(\psi_i \chi_j + \psi_j \chi_i) - \tilde{c}^1 \chi_i \chi_j - \tilde{c}^2 \psi_i \psi_j). \quad (3.65)$$

The coboundary operator  $s_{\text{gh}}$  acts on  $\vartheta_i$  and  $\Theta_{ij}$  according to

$$s_{\text{gh}} \vartheta_i^\alpha = i \sum_{j=1}^N (\xi_i \cdot \xi_j) \xi_j^\alpha, \quad s_{\text{gh}} \Theta_{ij} = -i \sum_{k=1}^N (\xi_i \cdot \xi_k)(\xi_j \cdot \xi_k). \quad (3.66)$$

Equations (3.66) can be easily verified explicitly in the spinor representation (3.1) using equations (3.64) and (3.65). The validity of equations (3.66) in the particular spinor representation implies their validity in any spinor representation equivalent to the particular spinor representation owing to their  $\mathfrak{so}(t, 3-t)$ -covariance.<sup>2</sup>

Equations (3.64) and (3.65) show that the various ghost polynomials involving the translations ghosts which appear in lemmas 3.1 and 3.2 are proportional to ghost polynomials (3.62) expressed in the spinor representation (3.1) respectively. Using additionally  $\psi_i = \xi_i^1$ ,  $i\chi_i = \xi_i^2$  and that  $\psi_i \chi_j - \chi_i \psi_j$  is the  $\mathfrak{so}(t, 3-t)$ -invariant product  $\xi_i \cdot \xi_j$  of  $\xi_i$  and  $\xi_j$  we can rewrite lemmas 3.1, 3.2 and 3.3 in an  $\mathfrak{so}(t, 3-t)$ -covariant form which extends them to all spinor representations equivalent to the spinor representation (3.1).

Lemma 3.1 yields:

**Lemma 3.4** ( $H_{\text{gh}}(s_{\text{gh}})$  for  $N = 1$ ).

*In the case  $N = 1$  a complete set of independent cohomology classes of  $H_{\text{gh}}(s_{\text{gh}})$  is  $\{[1], [\xi_1^1], [\xi_1^2], [\vartheta_1^1], [\vartheta_1^2], [\Theta_{11}]\}$  with  $\vartheta_1^\alpha$  and  $\Theta_{11}$  as in equations (3.62):*

$$s_{\text{gh}} \omega = 0 \Leftrightarrow \omega \sim a + \xi_1^\alpha a_\alpha + \vartheta_1^\alpha b_\alpha + \Theta_{11} b; \quad (3.67)$$

$$a + \xi_1^\alpha a_\alpha + \vartheta_1^\alpha b_\alpha + \Theta_{11} b \sim 0 \Leftrightarrow a = a_\alpha = b_\alpha = b = 0 \quad (3.68)$$

where  $a, a_\alpha, b_\alpha, b \in \mathbb{C}$ .

---

<sup>2</sup>Alternatively one may verify equations (3.66) directly in a spinor representation independent manner using the "completeness relation"  $\delta_{\alpha\beta}^\gamma \delta_{\gamma\delta}^\alpha + \Gamma_{\alpha\beta}^\gamma \Gamma_{\gamma\delta}^\alpha = 2\delta_{\alpha\beta}^\gamma \delta_{\gamma\delta}^\alpha$  of the gamma-matrices in  $D = 3$  dimensions.



**Comment:** In the case  $N = 1$  equations (3.66) give  $s_{\text{gh}}\vartheta_1^\alpha = 0$  and  $s_{\text{gh}}\Theta_{11} = 0$  owing to  $\xi_1 \cdot \xi_1 = 0$  (one has  $\xi_i \cdot \xi_j = -\xi_j \cdot \xi_i$  in the present case). Note that lemma 3.4 only applies to signatures (1, 2) and (2, 1) because  $N$  is even for signatures (3, 0) and (0, 3).

Lemma 3.2 yields:

**Lemma 3.5** ( $H_{\text{gh}}(s_{\text{gh}})$  for  $N = 2$ ).

*In the case  $N = 2$*

(i) any cocycle in  $H_{\text{gh}}(s_{\text{gh}})$  can be written as a polynomial in the components of the supersymmetry ghosts  $\xi_1, \xi_2$  which is at most linear in the components of  $\xi_1$ , or as polynomials in the components of  $\xi_2$  times  $\Theta_{12}$  or  $\Theta_{11} - \Theta_{22}$  with  $\Theta_{ij}$  as in equations (3.62):

$$s_{\text{gh}}\omega = 0 \Leftrightarrow \omega \sim a(\xi_2) + \xi_1^\alpha a_\alpha(\xi_2) + \Theta_{12}b_1(\xi_2) + (\Theta_{11} - \Theta_{22})b_2(\xi_2) \quad (3.69)$$

where  $a(\xi_2)$ ,  $a_\alpha(\xi_2)$ ,  $b_1(\xi_2)$  and  $b_2(\xi_2)$  are polynomials in the components of  $\xi_2$ ;

(ii) a cocycle  $a(\xi_2) + \xi_1^\alpha a_\alpha(\xi_2) + \Theta_{12}b_1(\xi_2) + (\Theta_{11} - \Theta_{22})b_2(\xi_2)$  is a coboundary in  $H_{\text{gh}}(s_{\text{gh}})$  if and only if  $a$ ,  $b_1$  and  $b_2$  vanish and if  $\xi_1^\alpha a_\alpha(\xi_2)$  depends on  $\xi_1$  only via the  $\mathfrak{so}(t, 3-t)$ -invariant product  $\xi_1 \cdot \xi_2$  and at least quadratically on the components of  $\xi_2$ :

$$\begin{aligned} a(\xi_2) + \xi_1^\alpha a_\alpha(\xi_2) + \Theta_{12}b_1(\xi_2) + (\Theta_{11} - \Theta_{22})b_2(\xi_2) &\sim 0 \Leftrightarrow \\ a = b_1 = b_2 = 0 \wedge \xi_1^\alpha a_\alpha(\xi_2) &= (\xi_1 \cdot \xi_2) \xi_2^\alpha d_\alpha(\xi_2) \end{aligned} \quad (3.70)$$

for some polynomials  $d_\alpha(\xi_2)$  in the components of  $\xi_2$ .

**Comment:** In the case  $N = 2$  equations (3.66) yield  $s_{\text{gh}}\vartheta_1^\alpha = i(\xi_1 \cdot \xi_2)\xi_2^\alpha$ ,  $s_{\text{gh}}\Theta_{12} = 0$  and  $s_{\text{gh}}\Theta_{11} = s_{\text{gh}}\Theta_{22} = -i(\xi_1 \cdot \xi_2)^2$ . These relations are behind the results that in (3.69) there is no analog of the term  $\vartheta_1^\alpha b_\alpha$  in lemma 3.4, that  $(\xi_1 \cdot \xi_2) \xi_2^\alpha d_\alpha(\xi_2)$  is a coboundary in  $H_{\text{gh}}(s_{\text{gh}})$  and that  $\Theta_{12}$  and  $\Theta_{11} - \Theta_{22}$  are cocycles in  $H_{\text{gh}}(s_{\text{gh}})$  for  $N = 2$ .

Lemma 3.3 yields:

**Lemma 3.6** ( $H_{\text{gh}}(s_{\text{gh}})$  for  $N > 2$ ).

*In the cases  $N > 2$*

(i) any cocycle in  $H_{\text{gh}}(s_{\text{gh}})$  can be written as a polynomial in the components of the supersymmetry ghosts  $\xi_1, \dots, \xi_N$  which is at most linear in the components of  $\xi_1$ :

$$s_{\text{gh}}\omega = 0 \Leftrightarrow \omega \sim a(\xi_2, \dots, \xi_N) + \xi_1^\alpha a_\alpha(\xi_2, \dots, \xi_N) \quad (3.71)$$

where  $a(\xi_2, \dots, \xi_N)$  and  $a_\alpha(\xi_2, \dots, \xi_N)$  are polynomials in the components of  $\xi_2, \dots, \xi_N$ ;

(ii) a cocycle  $a(\xi_2, \dots, \xi_N) + \xi_1^\alpha a_\alpha(\xi_2, \dots, \xi_N)$  is a coboundary in  $H_{\text{gh}}(s_{\text{gh}})$  if and only if  $a(\xi_2, \dots, \xi_N)$  is proportional to the sum of all squared  $\mathfrak{so}(t, 3-t)$ -invariants  $\xi_i \cdot \xi_j$  with  $i, j \in \{2, \dots, N\}$  and if  $\xi_1^\alpha a_\alpha(\xi_2, \dots, \xi_N)$  depends on  $\xi_1$  only via  $\sum_{i=2}^N (\xi_1 \cdot \xi_i) \xi_i^1$  or  $\sum_{i=2}^N (\xi_1 \cdot \xi_i) \xi_i^2$ :

$$a(\xi_2, \dots, \xi_N) + \xi_1^\alpha a_\alpha(\xi_2, \dots, \xi_N) \sim 0 \Leftrightarrow$$

$$\begin{aligned}
a(\xi_2, \dots, \xi_N) &= \frac{1}{2} \sum_{i=2}^N \sum_{j=2}^N (\xi_i \cdot \xi_j)^2 d(\xi_2, \dots, \xi_N) \wedge \\
\xi_1^\alpha a_{\underline{\alpha}}(\xi_2, \dots, \xi_N) &= \sum_{i=2}^N (\xi_1 \cdot \xi_i) \xi_i^\alpha d_{\underline{\alpha}}(\xi_2, \dots, \xi_N)
\end{aligned} \tag{3.72}$$

for some polynomials  $d(\xi_2, \dots, \xi_N)$  and  $d_{\underline{\alpha}}(\xi_2, \dots, \xi_N)$  in the components of  $\xi_2, \dots, \xi_N$ .

**Comment:** Equations (3.66) imply

$$s_{\text{gh}}(\Theta_{11} - \sum_{i=2}^N \Theta_{ii}) = i \sum_{i=2}^N \sum_{j=2}^N (\xi_i \cdot \xi_j)^2.$$

This is behind the condition on  $a(\xi_2, \dots, \xi_N)$  in (3.72). Using in addition the first equation (3.66), one can reformulate (3.72) according to:

$$\begin{aligned}
a(\xi_2, \dots, \xi_N) + \xi_1^\alpha a_{\underline{\alpha}}(\xi_2, \dots, \xi_N) &\sim 0 \Leftrightarrow \\
a(\xi_2, \dots, \xi_N) &= s_{\text{gh}}(\Theta_{11} - \sum_{i=2}^N \Theta_{ii}) d'(\xi_2, \dots, \xi_N) \wedge \\
\xi_1^\alpha a_{\underline{\alpha}}(\xi_2, \dots, \xi_N) &= s_{\text{gh}} \vartheta_1^\alpha d'_{\underline{\alpha}}(\xi_2, \dots, \xi_N)
\end{aligned} \tag{3.73}$$

for some polynomials  $d'(\xi_2, \dots, \xi_N)$  and  $d'_{\underline{\alpha}}(\xi_2, \dots, \xi_N)$  in the components of  $\xi_2, \dots, \xi_N$ .

## 4 Conclusion

We have computed the primitive elements of the supersymmetry algebra cohomology for supersymmetry algebras (1.1) in  $D = 2$  and  $D = 3$  dimensions for all signatures  $(t, D - t)$  and all numbers  $N$  of sets of Majorana type supersymmetries (depending on the particular dimension and signature, these are Majorana-Weyl, Majorana or symplectic Majorana supersymmetries).

Thereby we have introduced methods which are applicable and useful also for analogous computations in higher dimensions. These are:

- "dimension-climbing", i.e. using the results in a lower dimension to derive the results in a higher dimension, cf. sections 2.1 and 3.1.1;
- "ghost-matching" for different signatures, i.e. using appropriately defined ghost variables that allow one to match transformations and results for different signatures in a particular dimension, cf. sections 2.3 and 3.1 (see also section 5.3 of [1]);
- "covariantization" of results, i.e. rewriting the results obtained in a particular spinor representation in an  $\mathfrak{so}(t, D - t)$ -covariant way so that they become valid for any other equivalent spinor representation, cf. sections 2.2 and 3.2.

Furthermore the results exhibit features that will be met also in higher dimensions and are typical for the supersymmetry algebra cohomology. These are:

- the dependence of the primitive elements on the translation ghosts via  $\mathfrak{so}(t, D-t)$ -covariant ghost polynomials as in equations (3.62), cf. lemmas 3.4 and 3.5;
- a decrease of the maximal  $c$ -degree (= degree in the translation ghosts) of primitive elements for increasing  $N$  (in  $D = 3$  dimensions there are primitive elements with  $c$ -degrees zero and one in the cases  $N = 1$  and  $N = 2$  but in the cases  $N > 2$  all primitive elements have  $c$ -degree zero, cf. lemmas 3.1, 3.2 and 3.3).

As a final remark we add that the results for  $D = 2$  apply analogously also to the case with an alternative choice of the charge conjugation matrix  $C$  ( $\sigma_1$  in place of  $\sigma_2$  in the particular spinor representations given in equations (2.1), (2.18) and (2.19)): in lemma 2.3 one just has to substitute  $\xi_{1+}^{+\alpha}\xi_{1-}^{-\beta}C_{\underline{\alpha}\underline{\beta}}^{-1}$  for  $\xi_{1+}^{+\alpha}\xi_{1-}^{-\beta}(\hat{\Gamma}C^{-1})_{\underline{\alpha}\underline{\beta}}$  and in lemma 2.5  $\xi_1^{\underline{\alpha}}\xi_1^{\underline{\beta}}C_{\underline{\alpha}\underline{\beta}}^{-1}$  for  $\xi_1^{\underline{\alpha}}\xi_1^{\underline{\beta}}(\hat{\Gamma}C^{-1})_{\underline{\alpha}\underline{\beta}}$  (cf. section 5.5 of [1]).

## References

- [1] F. Brandt, “Supersymmetry algebra cohomology I: Definition and general structure,” arXiv:0911.2118 [hep-th].